My primary research interest is in arithmetic geometry, specifically in motives and motivic zeta functions. I am particularly interested in the relationship between zeta functions and the big ring of Witt vectors. A recurrent theme in number theory is that the special values of certain zeta functions, which are analytically defined objects, can be expressed in terms of algebraic invariants. These algebraic invariants (\(\ell\)-adic cohomology, Weil-\(\acute{e}\)tale cohomology, . . .) satisfy usual properties, for example they are typically additive and multiplicative (via a Künneth formula, for instance). We call such invariants Euler-Poincaré characteristics on the category of certain spaces (varieties over \(\text{Spec } k\), schemes of finite type over \(\text{Spec } \mathbb{Z}\), . . .). If we view the zeta function as a map on the category of such spaces, then it is natural to ask if zeta functions themselves can be considered Euler-Poincaré characteristics. This would shed light on why the special values of zeta functions take the form of Euler characteristic formulas: the zeta function is itself an Euler characteristic.

My research centers on understanding zeta functions as Euler-Poincaré characteristics taking values in the big Witt ring \(W_R\).

1. Exponentiation of motivic zeta functions

Let \(k\) be a field and \(\text{Var}_k\) be the category of varieties over \(k\) (reduced schemes of finite type over \(\text{Spec } k\)). The Grothendieck ring of varieties \(K_0(\text{Var}_k)\) is the abelian group generated by symbols \([X]\) of isomorphism classes of \(X \in \text{Var}_k\) subject to the scissor relation \([X] = [Y] + [X \setminus Y]\) for \(Y\) any closed subvariety of \(X\). \(K_0(\text{Var}_k)\) is a commutative ring under the product \([X] \cdot [Y] = [X \times_k Y]\). Given a commutative ring with identity \(R\), a motivic measure is a ring homomorphism on the Grothendieck ring \(K_0(\text{Var}_k)\). Associated to a motivic measure \(\mu : K_0(\text{Var}_k) \to R\), Kapranov in [16] constructs a motivic zeta function as follows: for \(X\) quasi-projective, it is the generating series for (the measure of) symmetric powers \(\text{Sym}^n X\):

\[
\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu(\text{Sym}^n X)t^n \in R[[t]].
\]

This extends to a group homomorphism on \(K_0(\text{Var}_k)\) taking values in the group of invertible power series \((1 + tR[[t]], \times)\).

As the Grothendieck ring \(K_0(\text{Var}_k)\) is indeed a ring, we may study the (possible) structure of \(\zeta_\mu\) as a ring homomorphism. That is, we endeavor to describe the product \(\zeta_\mu(X \times_k Y)\) using a product structure on \((1 + tR[[t]], \times)\). As it turns out, the correct product structure is often given by the big Witt ring \(W(R)\), denoted \(*_W\). Following Ramachandran [24], a motivic measure can be exponentiated if the associated motivic zeta function is a ring homomorphism taking values in the Witt ring, \(\zeta_\mu : K_0(\text{Var}_k) \to W(R)\).

For example:

- In the finite field case, the classical Hasse-Weil zeta function \(Z(X, t)\) is the motivic zeta function \(\zeta_{\#\mu}(X)\) associated to the counting measure \(\mu_{\#}(X) = \#X(\mathbb{F}_q)\) on \(X \in \text{Var}_{\mathbb{F}_q}\). It is well known that the zeta function of a product of varieties \(Z(X \times_k Y, t)\) is given by the Witt product \(Z(X, t) *_W Z(Y, t)\); hence \(\mu_{\#}\) can be exponentiated.
- In the char \(k = 0\) case, associated to the symmetric monoidal functor \(h : \text{SmProj}(k) \to \text{Chow}(k)\)\(_\mathbb{Q}\)}

\(^{1}\)For char \(k > 0\), one must also mod out by surjective radicial morphisms (see [24]).
Gillet and Soulé [7] construct a measure $\mu_{GS}$ taking values in $K_0(\text{Chow}(k)_\Q)$. This measure $\mu_{GS}$, and thus all measures factoring through it, can be exponentiated [20, Prop. 4.8].

- Larsen and Lunts construct a measure $\mu_{LL}$ such that for a certain product of curves $C_1 \times C_2$, the motivic zeta function $\zeta_{\mu_{LL}}(C_1 \times C_2, t)$ is not rational [13, Theorem 7.6]. This implies that the universal motivic zeta function $\zeta_{\mu}$ cannot be exponentiated since a Witt product of rational functions is rational, and $\zeta_{\mu}(C)$ is rational for all curves $C$ [10].

Although Witt-type phenomena of zeta functions appears in prior studies, identifying the Kapranov motivic zeta function as an object in the big Witt ring is only earnestly present in Ramachandran [24] and Ramachandran-Tabuada [26].

If $\mu$ can be exponentiated, its associated zeta function is a ring homomorphism into $W(R)$, and thus itself defines a motivic measure. This measure is the \textit{induced zeta function measure} $\mu_Z(X) = \zeta_{\mu}(X,t)$ and its associated motivic zeta function $\zeta_{\mu_Z}$ is a generating series for the zeta function of symmetric powers

$$\zeta_{\mu_Z}(X,u) = \sum_{n=0}^{\infty} Z(\text{Sym}^n, t) u^n \in 1 + uW(R)[[u]].$$

We may ask if this zeta function measure $\mu_Z$ can be exponentiated, i.e. if $\zeta_{\mu_Z}$ takes values in $W(W(R))$. In my thesis [15], I study several such questions related to induced zeta function measures posed in [24] and [26].

1.1. The finite field case. In the case of $\text{Var}_{\F_q}$, the Hasse-Weil zeta function may be realized as $Z(X,t) = \prod_i P_i(t)^{-1 \alpha_i + 1}$ with $P_i(t) = \prod_j (1 - \alpha_{ij} t)^{-1}$ where the $\alpha_{ij}$ are (inverse) eigenvalues of the Frobenius $\Phi$ acting on $H^i_{et,c}(X; \Q_{\ell})$. When written in the Witt ring $W(\Z)$, this takes the form of an Euler-Poincaré characteristic as an alternating sum of \textit{Teichmüller elements} $[\alpha]$:

$$Z(X,t) = \sum_{i,j} (-1)^{i+1} [\alpha_{ij}] \in W(\Q_{\ell}).$$

Thus the zeta function $Z(X,t)$ is a motivic measure $\mu_Z$ taking values in the ring $A = W(\Z)$; now consider its associated motivic zeta function $\zeta_{\mu_Z}$. We show that $\mu_Z$ exponentiates and $\zeta_{\mu_Z}$ takes values in $W(A) = W(W(\Z))$ the double Witt ring. To achieve this, we provide a similar formula

**Theorem 1** (H.). \textit{The Weil zeta function of a variety $X$ over $\F_q$, when considered as a motivic measure $\mu_Z(X) = Z(X,t)$, has an associated motivic zeta function $\zeta_{\mu_Z}$ taking the form}

$$\zeta_{\mu_Z}(X,u) = \sum_{i,j} (-1)^{i+1} [\left[ \alpha_{ij} \right] \right]$$

\textit{where the sum is taking place in the Witt ring $W(W(\Q_{\ell}))$.}

This formula can be used to compute the zeta function $Z(\text{Sym}^n X,t)$ of the symmetric power of a variety $X$ in terms of $Z(X,t)$. Note that the case of symmetric powers of smooth projective curves was worked out by MacDonald in [19]. Our results generalize and reframe this in the setting of motivic zeta functions, and can be considered a MacDonal-type formula for the zeta function of varieties over finite fields [14].

1.2. Lambda-ring measures. An important case of motivic zeta functions is where $\mu : K_0(\text{Var}_k) \to R$ takes values in a $\lambda$-ring $R$. For example, $\Z$ is (uniquely) a $\lambda$-ring; for $R$ a commutative ring with identity, the Grothendieck group $K_0(R)$ is a $\lambda$-ring; for $C$ a symmetric monoidal additive pseudo-abelian category, $K_0(C)$ is a $\lambda$-ring (see Heinloth [12]). In such cases, Ramachandran and Tabuada [26] provide a condition whereby $\mu$ can be exponentiated. The importance of $\lambda$-ring-valued motivic measures has been observed in Larsen and Lunts [13], Heinloth [12, 14], Gusein-Zade, Lluengo, and Melle-Hernandez [11, 10], del Baño Rollin [5, 6], and Maxim and Schurmann [20]. Our main focus is on motivic measures taking values in $R$ a $\lambda$-ring; our main observation is that for any ring $A$, the big Witt ring $W(A)$ is a $\lambda$-ring.
Let $R$ be a $\lambda$-ring and $\mu : K_0(\text{Var}_k) \to R$ a motivic measure. In this case, we have an opposite $\lambda$-ring map $\sigma : R \to W(R)$. While the Grothendieck ring of varieties is not known to have a $\lambda$-ring structure in general, it is a pre-$\lambda$-ring via the universal (Kapranov) motivic zeta function. A motivic measure $\mu$ is called a $\lambda$-ring measure $\mu : K_0(\text{Var}_k) \to R$ if it is a pre-$\lambda$-ring map. In [26], it is shown that such measures can be exponentiated, and

$$K_0(\text{Var}_k) \xrightarrow{\mu} R \xrightarrow{\sigma} W(R)$$

commutes. The case of the zeta function of varieties over finite fields may be restated in this setting using the opposite $\lambda$-ring structure map on $W(Z)$.

Given a $\lambda$-ring measure $\mu$, consider the induced motivic zeta measure $\mu_Z = \zeta_{\mu}$. This takes values in $W(R)$, which is has a natural $\lambda$-ring structure. We show that for $\lambda$-ring measures $\mu$, the induced motivic zeta measure $\mu_Z$ can always be exponentiated.

**Theorem 2** (H.). Let $\mu : K_0(\text{Var}_k) \to F$ be a $\lambda$-ring measure and $\mu_Z : K_0(\text{Var}_k) \to W(F)$ its induced motivic zeta function measure is also a $\lambda$-ring measure:

\[
\zeta_{\mu_Z}(X, u) = \sum_n \zeta_{\mu}(\text{Sym}^n X, t) u^n = \sigma_n(\zeta_{\mu}(X, t))
\]

and thus $\mu_Z$ can be exponentiated with $\zeta_{\mu_Z}$ taking values in $W(A) = W(W(R))$.

Moreover, this process may be iterated ad infinitum, i.e. $\zeta_{\mu_Z}$ is itself a $\lambda$-ring measure. These results follow from the $\lambda$-ring structure of the Witt ring $W(R)$, particularly in the case where $R$ itself is a $\lambda$-ring; essentially, in these cases exponentiation always occurs via the opposite $\lambda$-ring structure map $\sigma_n$.

2. **Ongoing and Future Research Projects**

We begin by discussing some possible extensions of the above results and immediate questions arising therefrom.

2.1. **Lambda-ring measures versus the counting measure.** The zeta function for varieties over finite fields $Z(X, t)$ is the Kapranov motivic zeta function associated to the counting measure $\mu_\#$ on $\text{Var}_F$. We have shown that $Z(X, t)$, as a measure, can be exponentiated; however, counting measure is not a $\lambda$-ring measure. There is a unique $\lambda$-ring structure on $Z$: for $a \in Z$,

\[
\lambda_\#(a) = (1 + t)^a, \quad \lambda^n_\#(a) = \binom{a}{n}, \quad \sigma^n_\#(a) = \binom{a + n - 1}{n}.
\]

It is easy to verify that $\mu_\#(\text{Sym}^n X) \neq \sigma^n_\#(\mu_\#(X))$. For instance, $X = \mathbb{A}^1$, $\text{Sym}^n \mathbb{A}^1 = \mathbb{A}^n$ and $\mu_\#(\mathbb{A}^1) = q$ whereas $\mu_\#(\mathbb{A}^n) = q^n \neq \sigma^n_\#(q)$. As this is the only $\lambda$-ring structure on $Z$, counting measure can not be a pre-$\lambda$-ring map.

The zeta function measure $\mu_Z = Z(X, t)$ is a $\lambda$-ring measure while $\mu_\#$ is not. This should be possible for other abstract measures as well.

**Question 1.** When and how do motivic zeta functions arise from measures $\mu : K_0(\text{Var}_k) \to R$ which are not $\lambda$-ring measures? Can the induced zeta measures be exponentiated?
2.2. *K*-theory of endomorphisms. The big Witt ring $W(R)$ is related to the $K$-theory of endomorphisms. Consider $\text{End}_R$ the category of pairs $(P, f)$ where $P$ is a finitely generated projective $R$-module and $f : P \to P$ is an endomorphism, with morphisms $\varphi : (P, f) \to (P', f')$ respecting the endomorphisms $f$ and $f'$. The group $K_0(\text{End}_R)$ is a $\lambda$-ring, and there is an injective ring homomorphism due to Almkvist [1] (see also Grayson [8])

$$K_0(\text{End}_R)/J \hookrightarrow W(R), \quad (P, f) \mapsto \det(\text{id}_P - tf)$$

where $J$ is the ideal generated by $(R, 0)$ in $K_0(\text{End}_R)$.

We wish to explore how this relates to exponentiation of motivic measures and motivic zeta functions. For instance,

**Question 2.** For which measures $\mu$ does $\zeta_\mu$ take values in the image $K_0(\text{End}_R)/J$ in $W(R)$? In such cases, can the induced zeta function measure $\mu_2$ be exponentiated?

2.2.1. Hilbert Scheme of points. Given a complex smooth quasi-projective variety of dimension $d$, consider $\text{Hilb}^n_X$, the Hilbert scheme of zero-dimensional subschemes of $X$ of length $n$. Given the class $[\text{Hilb}^n_X]$ in $K_0(\text{Var}_k)$, we can form the Hilbert scheme zeta function

$$H_X(t) = 1 + \sum_{n \geq 1} [\text{Hilb}^n_X]t^n$$

in $\Lambda(K_0(\text{Var}_k))$. We would like to better understand the product structure of such Hilbert scheme zeta functions. In particular, what type of multiplicative structure, analogous to the Witt ring $W(K_0(\text{Var}_k))$ is required for $\text{Hilb}_X$?

2.3. Other related future research projects. The main focus of my research has thus far centered on the $\lambda$-ring structure of the big Witt ring $W(R)$ and motivic zeta functions taking values therein. I hope to expand my research to include Witt-type phenomena in other incidences of zeta functions: Lichtenbaum’s Weil-étale Euler characteristic interpretation of special values of arithmetic zeta functions, zeta interpretations of analytic torsion and Reidemeister-Franz torsion in arithmetic topology, and various appearances of zeta elements in $K$-theory and $K(1)$-local spectra. Here I briefly mention a few specific future research projects.

Higher Euler characteristics. Following Ramachandran [25], another invariant which shows up in the study of special values of zeta functions, as well as in many other fields, is the so-called secondary Euler characteristic $\chi'$. Given a finite chain complex $C_*$, the usual Euler characteristic is defined as an alternating sum of ranks $\chi(C_*) = \sum_i (-1)^i \text{rank } C_i$; when defined on spaces $X$, its universal value group is $K_0(\text{Var}_k)$. If $\chi(C_*) = 0$, then we may define the secondary Euler characteristic

$$\chi'(C_*) = \sum_i (-1)^i \text{rank } C_i.$$

This invariant appears in the definition of Ray-Singer analytic torsion [27], in Milne’s correcting factor for special values of zeta functions [21], and is related to the Adams operations on higher $K$-groups (see [9]). Notice that while $\chi'$ is additive, it is not multiplicative. We wish to study the analogous universal value group of secondary Euler characteristics and possible relations to motivic zeta functions and their special values.

Grothendieck spectrum of varieties. The Grothendieck ring of varieties $K_0(\text{Var}_k)$, arising from cut-and-paste scissors relations on $\text{Var}_k$, was extended by Zakharevich [28] and Campbell [3] to a $K$-theory spectrum of varieties $K(\text{Var}_k)$. A recent preprint of Campbell, Wolfson and Zakharevich [1] takes the further step of lifting the Hasse-Weil zeta function on $\text{Var}_k$ to a map of $K$-theory spectra. Moreover, recent results of Blumberg, Gepner and Tabuada [2] extend the endomorphism functor $\text{End}(-)$ to (small, idempotent-complete stable) $\infty$-categories. I hope to explore motivic measures and motivic zeta functions in this setting of the Grothendieck spectrum of varieties. Specifically, we may ask in what sense is the map on $K$-theory spectra can be exponentiated, and if the induced motivic zeta function measure also lifts to a map of ring spectra.
Research Statement

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Arithmetic topology. One interesting relationship between topology and number theory is the analogy between knots and primes (sometimes called arithmetic topology or the MKR dictionary (Mazur-Kapranov-Reznokov) (see [22]). Its central observation is that circles $S^1$ with $\pi_1(S^1) = \mathbb{Z}$ have an étale homotopic analog in primes $\text{Spec} \mathbb{F}_p$ with $\pi_1^e(\text{Spec} \mathbb{F}_p) = \hat{\mathbb{Z}}$. Thus given an orientable, closed 3-manifold $M$ and a ring of integers $O_k$ of a number field $k$, knots $S^1 \hookrightarrow M$ are analogous to primes $\text{Spec} \mathbb{F}_p \hookrightarrow \text{Spec} O_k$. One of the ideas that arises from this is the analogy between the Ray-Singer conjecture on topological torsion invariants and the main conjecture in Iwasawa theory; an infinite cyclic covering $\overline{M}/M$ corresponds to cyclotomic $\mathbb{Z}_p$-extensions $k_\infty/k$. I am interested in exploring these concepts further; specifically in the context of the determinant functor ETNC formulation of the Iwasawa main conjecture and Kurihara’s refined main conjecture on higher Fitting ideals (see [17]).

References

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