My primary research interest is in arithmetic geometry, specifically in motives and motivic zeta functions. I am particularly interested in the relationship between zeta functions and the big ring of Witt vectors. A recurrent theme in number theory is that the special values of zeta functions, which are analytically defined objects, can be expressed in terms of algebraic invariants. These algebraic invariants ($\ell$-adic cohomology, algebraic $K$-groups, Weil-étale cohomology, ... ) satisfy usual properties, for example they are typically additive and multiplicative (via a Künneth formula, for instance). We call such invariants Euler-Poincaré characteristics on the category of certain spaces (varieties over $\text{Spec} \ k$, schemes of finite type over $\text{Spec} \ Z$, ... ). If we view the zeta function as a map on the category of such spaces, then it is natural to ask if zeta functions themselves can be considered Euler-Poincaré characteristics. This would shed light on why the special values of zeta functions take the form of Euler characteristic formulas: the zeta function is itself an Euler characteristic. My research centers on understanding zeta functions as Euler-Poincaré characteristics taking values in the big Witt ring $W(R)$.

1. Exponentiation of motivic zeta functions

Let $k$ be a field and $\text{Var}_k$ be the category of varieties over $k$ (reduced schemes of finite type over $\text{Spec} \ k$). The Grothendieck ring of varieties $K_0(\text{Var}_k)$ is the abelian group generated by symbols $[X]$ of isomorphism classes of $X \in \text{Var}_k$ subject to the scissor relation $[X] = [Y] + [X \setminus Y]$ for $Y$ any closed subvariety of $X$.

$K_0(\text{Var}_k)$ is a commutative ring under the product $[X] \cdot [Y] = [X \times_k Y]$. Given a commutative ring with identity $R$, a motivic measure is a ring homomorphism on the Grothendieck ring $K_0(\text{Var}_k)$. Associated to a motivic measure $\mu : K_0(\text{Var}_k) \to R$, Kapranov in [16] constructs a motivic zeta function as follows: for $X$ quasi-projective, it is the generating series for (the measure of) symmetric powers $\text{Sym}^n X$:

$$\zeta_{\mu}(X,t) = \sum_{n=0}^{\infty} \mu(\text{Sym}^n X)t^n \in R[[t]].$$

This extends to a group homomorphism on $K_0(\text{Var}_k)$ taking values in the group of invertible power series $(1 + tR[[t]], \times)$. Some standard questions arise regarding $\zeta_{\mu}$; namely, does it satisfy Weil-conjecture type conditions? Is it a rational function? Does it satisfy a function equation? What characterizes its special values?

As the Grothendieck ring $K_0(\text{Var}_k)$ is indeed a ring, we may study the (possible) structure of $\zeta_{\mu}$ as a ring homomorphism. That is, we endeavor to describe the product $\zeta_{\mu}(X \times_k Y)$ using a product structure on $(1 + tR[[t]], \times)$. As it turns out, the correct product structure is often given by the big Witt ring $W(R)$, denoted $\ast_W$. Following Ramachandran [23], a motivic measure can be exponentiated if the associated motivic zeta function is a ring homomorphism taking values in the Witt ring, $\zeta_{\mu} : K_0(\text{Var}_k) \to W(R)$.

For example:

- In the finite field case, the classical Hasse-Weil zeta function $Z(X,t)$ is the motivic zeta function $\zeta_{\mu_{\#}}$ associated to the counting measure $\mu_{\#}(X) = \#X(\mathbb{F}_q)$ on $X \in \text{Var}_{\mathbb{F}_q}$. The zeta function of a product of varieties $Z(X \times_k Y, t)$ is given by the Witt product $Z(X,t) \ast_W Z(Y,t)$; hence $\mu_{\#}$ can be exponentiated (see [24]).
• In the char $k = 0$ case, associated to the symmetric monoidal functor

$$h : SmProj(k) \longrightarrow \text{Chow}(k)_\mathbb{Q}$$

Gillet and Soulé [7] construct a measure $\mu_{GS}$ taking values in $K_0(\text{Chow}(k)_\mathbb{Q})$. This measure $\mu_{GS}$, and thus all measures factoring through it, can be exponentiated [20 Prop. 4.8].

• Larsen and Lunts construct a measure $\mu_{LL}$ such that for a certain product of curves $C_1 \times C_2$, the motivic zeta function $\zeta_{\mu_{LL}}(C_1 \times C_2, t)$ is not rational [18 Theorem 7.6]. This implies that the universal motivic zeta function $\zeta_\alpha$ cannot be exponentiated since a Witt product of rational functions is rational., and $\zeta_{\mu}(C)$ is rational for all curves $C$ [19].

Although Witt-type phenomena of zeta functions appears in prior studies, identifying the Kapranov motivic zeta function as (potentially) an object in the big Witt ring is only earnestly present in Ramachandran [24] and Ramachandran-Tabuada [26]. In [26], exponentiation of motivic measures is used to study the rationality of $\zeta_\alpha$. For example, it is shown that for $\mu_{GS}$, $\zeta_{\mu_{GS}}(X \times Y)$ is rational if $\zeta_{\mu_{GS}}(X)$ and $\zeta_{\mu_{GS}}(Y)$ are (as in the case of Kimura-finite motives).

If $\mu$ can be exponentiated, its associated zeta function is a ring homomorphism into $W(R)$, and thus itself defines a motivic measure. This measure is the induced zeta function measure $\mu_Z(X) = \zeta_{\mu}(X, t)$ and its associated motivic zeta function $\zeta_{\mu_Z}$ is a generating series for the zeta function of symmetric powers

$$\zeta_{\mu_Z}(X, u) = \sum_{n=0}^{\infty} Z(\text{Sym}^n, t) \ u^n \in 1 + uW(R)[[u]].$$

We may ask if this zeta function measure $\mu_Z$ can be exponentiated, i.e. if $\zeta_{\mu_Z}$ takes values in $W(W(R))$. In my thesis [15], I study several such questions related to induced zeta function measures posed in [24] and [26].

1.1. The finite field case. In the case of $\text{Var}_{\mathbb{F}_q}$, the Hasse-Weil zeta function may be realized as $Z(X, t) = \prod_i P_i(t)^{-1 + t^{\alpha_i}}$ with $P_i(t) = \prod_\ell (1 - \alpha_{ij}t)^{-1}$ where the $\alpha_{ij}$ are (inverse) eigenvalues of the Frobenius $\Phi$ acting on $H^1_{et,c}(\overline{X}; \mathbb{Q}_\ell)$. When written in the Witt ring $W(\mathbb{Z})$, this takes the form of an Euler-Poincaré characteristic as an alternating sum of Teichmüller elements $[\alpha]$:

$$Z(X, t) = \sum_{i,j} (-1)^{i+1} [\alpha_{ij}] \in W(\mathbb{Q}_\ell).$$

Thus the zeta function $Z(X, t)$ is a motivic measure $\mu_Z$ taking values in the ring $A = W(\mathbb{Z})$; now consider its associated motivic zeta function $\zeta_{\mu_Z}$. We show that $\mu_Z$ exponentiates and $\zeta_{\mu_Z}$ takes values in $W(A) = W(W(\mathbb{Z}))$ the double Witt ring. To achieve this, we provide a similar formula

**Theorem 1 (H.).** The Weil zeta function of a variety $X$ over $\mathbb{F}_q$, when considered as a motivic measure $\mu_Z(X) = Z(X, t)$, has an associated motivic zeta function $\zeta_{\mu_Z}$ taking the form

$$\zeta_{\mu_Z}(X, u) = \sum_{i,j} (-1)^{i+1} [[\alpha_{ij}]]$$

where the sum is taking place in the Witt ring $W(W(\mathbb{Q}_\ell))$.

This formula can be used to compute the zeta function $Z(\text{Sym}^n X, t)$ of the symmetric power of a variety $X$ in terms of $Z(X, t)$. Note that the case of symmetric powers of smooth projective curves was worked out by MacDonald in [19]. Our results generalize and reframe this in the setting of motivic zeta functions, and can be considered a MacDonald-type formula for the zeta function of varieties over finite fields [14].
1.2. **Lambda-ring measures.** An important case of motivic zeta functions is where \( \mu : K_0(\text{Var}_k) \to R \) takes values in a \( \lambda \)-ring \( R \). For example, \( \mathbb{Z} \) is (uniquely) a \( \lambda \)-ring; for \( R = \mathbb{R} \) a commutative ring with identity, the Grothendieck group \( K_0(R) \) is a \( \lambda \)-ring; for \( C \) a symmetric monoidal additive pseudo-abelian category, \( K_0(C) \) is a \( \lambda \)-ring (see Heinloth [12]). In such cases, Ramachandran and Tabuada [26] provide a condition whereby \( \mu \) can be exponentiated. The importance of \( \lambda \)-ring-valued motivic measures has been observed in Larsen and Lunts [15], Heinloth [12] [13], Gusein-Zade, Llengu, and Melle-Hernandez [11] [10], del Baño Rollin [5] [6], and Maxim and Schurmann [20].

Let \( R \) be a \( \lambda \)-ring and \( \mu : K_0(\text{Var}_k) \to R \) a motivic measure. In this case, we have an opposite \( \lambda \)-ring map \( \sigma_t : R \to W(R) \). While the Grothendieck ring of varieties is not known to have a \( \lambda \)-ring structure in general\(^2\), it is a pre-\( \lambda \)-ring via the universal (Kapranov) motivic zeta function [18]. That is, the class of \([\text{Sym}^n X]\) in \( K_0(\text{Var}_k) \) defines a pre-\( \lambda \)-ring structure. A motivic measure \( \mu \) is called a \( \lambda \)-ring measure \( \mu : K_0(\text{Var}_k) \to R \) if it is a pre-\( \lambda \)-ring map. Such measures can be exponentiated, and the diagram

\[
\begin{array}{ccc}
K_0(\text{Var}_k) & \xrightarrow{\mu} & R \\
\downarrow{\zeta_\mu} & & \downarrow{\sigma_t} \\
W(R) & & \\
\end{array}
\]

commutes.

Given a \( \lambda \)-ring measure \( \mu \), consider the induced motivic zeta measure \( \mu_\lambda = \zeta_\mu \). This takes values in \( W(R) \), which is has a natural \( \lambda \)-ring structure. We show that for \( \lambda \)-ring measures \( \mu \), the induced motivic zeta measure \( \mu_\lambda \) can always be exponentiated.

**Theorem 2 (H.).** Let \( \mu : K_0(\text{Var}_k) \to F \) be a \( \lambda \)-ring measure and \( \mu_\lambda : K_0(\text{Var}_k) \to W(F) \) its induced motivic zeta function measure is also a \( \lambda \)-ring measure:

\[
\zeta_{\mu_\lambda}(X, u) = \sum_n \zeta_\mu(\text{Sym}^n X, t) u^n = \sigma_u(\zeta_\mu(X, t))
\]

and thus \( \mu_\lambda \) can be exponentiated with \( \zeta_{\mu_\lambda} \) taking values in \( W(A) = W(W(R)) \).

Moreover, this process may be iterated ad infinitum, i.e. \( \zeta_{\mu_\lambda} \) is itself a \( \lambda \)-ring measure. These results follow from the \( \lambda \)-ring structure of the Witt ring \( W(R) \), particularly in the case where \( R \) itself is a \( \lambda \)-ring; essentially, in these cases exponentiation always occurs via the opposite \( \lambda \)-ring structure map \( \sigma_u \). The case of the zeta function of varieties over finite fields in Theorem 1 may be restated in this setting using the opposite \( \lambda \)-ring structure map on \( W(\mathbb{Z}) \).

2. **Ongoing and Future Research Projects**

We begin by discussing some extensions of the above results and immediate questions arising therefrom.

2.1. **Lambda-ring measures versus the counting measure.** The zeta function for varieties over finite fields \( Z(X, t) \) is the Kapranov motivic zeta function associated to the counting measure \( \mu_{\#} \) on \( \text{Var}_F \). We have shown that \( Z(X, t) \), as a measure, can be exponentiated; however, counting measure is not a \( \lambda \)-ring measure. There is a unique \( \lambda \)-ring structure on \( \mathbb{Z} \): for \( a \in \mathbb{Z} \),

\[
\lambda_a(t) = (1 + t)^a, \quad \lambda^n(a) = \binom{a}{n}, \quad \sigma^n(a) = \binom{a + n - 1}{n}.
\]

It is easy to verify that \( \mu_{\#}(\text{Sym}^n X) = \sigma^n(\mu_{\#}(X)) \). For instance, \( X = \mathbb{A}^1, \text{Sym}^n \mathbb{A}^1 = \mathbb{A}^n \) and \( \mu_{\#}(\mathbb{A}^1) = q \) whereas \( \mu_{\#}(\mathbb{A}^n) = q^n \neq \sigma^n(q) \). As this is the only \( \lambda \)-ring structure on \( \mathbb{Z} \), counting measure can not be a pre-\( \lambda \)-ring map.

\(^2\)some authors prefer to use the term special \( \lambda \)-ring.
The zeta function measure $\mu_Z = Z(X, t)$ is a $\lambda$-ring measure while $\mu_\#$ is not. This should be possible for other abstract measures as well.

**Question 1.** When and how do motivic zeta functions arise from measures $\mu : K_0(\text{Var}_k) \to R$ which are not $\lambda$-ring measures? Can the induced zeta measures be exponentiated?

2.2. **$K$-theory of endomorphisms.** The big Witt ring $W(R)$ is related to the $K$-theory of endomorphisms. Consider $\text{End}_R$ the category of pairs $(P, f)$ where $P$ is a finitely generated projective $R$-module and $f : P \to P$ is an endomorphism, with morphisms $\varphi : (P, f) \to (P', f')$ respecting the endomorphisms $f$ and $f'$. The group $K_0(\text{End}_R)$ is a $\lambda$-ring, and there is an injective ring homomorphism due to Almkvist [1] (see also Grayson [3])

$$K_0(\text{End}_R)/J \to W(R), (P, f) \mapsto \det(id_P - tf)$$

where $J$ is the ideal generated by $(R, 0)$ in $K_0(\text{End}_R)$.

We wish to explore how this relates to exponentiation of motivic measures and motivic zeta functions. For instance,

**Question 2.** For which measures $\mu$ does $\zeta_\mu$ take values in the image $K_0(\text{End}_R)/J$ in $W(R)$? In such cases, can the induced zeta function measure $\mu_Z$ be exponentiated?

2.2.1. **Hilbert Scheme of points.** Given a complex smooth quasi-projective variety of dimension $d$, consider $\text{Hilb}_X^n$, the Hilbert scheme of zero-dimensional subschemes of $X$ of length $n$. Given the class $[\text{Hilb}_X^n]$ in $K_0(\text{Var}_C)$, we can form the Hilbert-scheme zeta function

$$H_X(t) = 1 + \sum_{n \geq 1} [\text{Hilb}_X^n] t^n$$

in $\Lambda(K_0(\text{Var}_C))$. We would like to better understand the product structure of such Hilbert scheme zeta functions. In particular, what type of multiplicative structure, analogous to the Witt ring $W$, is required for $\text{Hilb}_X$?

2.3. **Other related future research projects.** I briefly mention here a few specific future research projects.

*Higher Euler characteristics.* Following Ramachandran [25], another invariant which shows up in the study of special values of zeta functions, as well as in many other fields, is the secondary Euler characteristic $\chi'$.

Given a finite chain complex $C_\ast$, the usual Euler characteristic is defined as an alternating sum of ranks $\chi(C_\ast) = \sum (-1)^i \text{rank } C_i$; when defined on spaces $X$, its universal value group is $K_0(\text{Var}_k)$. If $\chi(C_\ast) = 0$, then we may define the secondary Euler characteristic

$$\chi'(C_\ast) = \sum (-1)^i \text{rank } C_i.$$

This invariant appears in the definition of Ray-Singer analytic torsion [27], in Milne’s correcting factor for special values of zeta functions [21], and is related to the Adams operations on higher $K$-groups (see [9]). Notice that while $\chi'$ is additive, it is not multiplicative. I wish to study the analogous universal value group of secondary Euler characteristics and possible relations to motivic zeta functions and their special values.

*Grothendieck spectrum of varieties.* The Grothendieck ring of varieties $K_0(\text{Var}_k)$, arising from cut-and-paste scissor relations on $\text{Var}_k$, was extended by Zakharevich [28] and Campbell [3] to a $K$-theory spectrum of varieties $K(\text{Var}_k)$. A recent preprint of Campbell, Wolfson and Zakharevich [4] takes the further step of lifting the Hasse-Weil zeta function on $\text{Var}_{\mathbb{Z}}$ to a map of $K$-theory spectra. Moreover, recent results of Blumberg, Gepner and Tabuada [2] extend the endomorphism functor $\text{End}(-)$ to (small, idempotent-complete stable) $\infty$-categories. I hope to explore motivic measures and motivic zeta functions in this setting of the Grothendieck spectrum of varieties. Specifically, we may ask in what sense can the map on $K$-theory spectra be exponentiated, and if the induced motivic zeta function measure also lifts to a map of ring spectra.
Arithmetic topology. One interesting relationship between topology and number theory is the analogy between knots and primes (sometimes called arithmetic topology or the MKR dictionary (Mazur-Kapranov-Reznokov) (see [22]). Its central observation is that circles $S^1$ with $\pi_1(S^1) = \mathbb{Z}$ have an étale homotopic analog in primes $\text{Spec} \mathbb{F}_p$ with $\pi_1^\text{ét}(\text{Spec} \mathbb{F}_p) = \mathbb{Z}$. Thus given an orientable, closed 3-manifold $M$ and a ring of integers $\mathcal{O}_k$ of a number field $k$, knots $S^1 \hookrightarrow M$ are analogous to primes $\text{Spec} \mathbb{F}_p \hookrightarrow \text{Spec} \mathcal{O}_k$. One of the ideas that arises from this is the analogy between the Ray-Singer conjecture on topological torsion invariants and the main conjecture in Iwasawa theory; an infinite cyclic covering $\widetilde{M}/M$ corresponds to cyclotomic $\mathbb{Z}_p$-extensions $k_\infty/k$. I am interested in exploring these concepts further; specifically in the context of the determinant functor ETNC formulation of the Iwasawa main conjecture and Kurihara’s refined main conjecture on higher Fitting ideals (see [17]).

References

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